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On a Property of The Coincidence Function Associated with The Logarithmic Derivative of a Polynomial*

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A new property of the coincidence function $\beta(z)$ is indicated. This property is applied to strengthen the well-known theorem on the polar derivative of a polynomial and also to obtain a result about the location of the zeros of a linear combination of a polynomial and its derivative.

Throughout this note, $D(a, r)$ will denote the closed disk with radius r about the point a.

It is well known that if $P(z)$ is a polynomial of degree n whose zeros β_i $(i = 1, 2, ..., n)$ lie in the closed unit disk, then for any z lying in the exterior of that closed disk, the logarithmic derivative

$$L(z) = \frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z - \beta_i} \tag{1}$$

can be written in a simplified form

$$L(z) = \frac{n}{z - \beta(z)}, \tag{1'}$$

obtained by letting all the β_i equal to $\beta = \beta(z)$. Numerous results in the geometry of polynomials depend heavily on the properties of the function $\beta(z)$ (e.g., $|\beta(z)| \leqslant 1$).

It is our intention to prove an additional property of the function $\beta(z)$ under the hypothesis that some β_i lies on the unit circumference. For simplicity we shall assume that this β_i is 1 and that its multiplicity is k. By a simple transformation one can use our result for the general case, where the restrictions of the last two sentences are not imposed.

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Our proof makes use of a result of G. Julia, in a slightly generalized form due to J. Wolff and C. Carathéodory (see e.g. [1], Sections 89–93). Namely,

Let $f(z)$ be holomorphic, with $|f(z)| < 1$, in the disk $|z| < 1$, and suppose that there exists a sequence $z_1, z_2, \ldots$ such that

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} f(z_n) = 1 \tag{2}$$

and such that

$$\lim_{n\to\infty} \frac{1 - |f(z_n)|}{1 - |z_n|} = \alpha \tag{3}$$

exists and is finite. Then

$$\lim_{x\to 1^-} \frac{1 - |f(x)|}{1 - x} = \lim_{x\to 1^-} \frac{|1 - f(x)|}{1 - x} = \lim_{x\to 1^-} \frac{1 - f(x)}{1 - x} = \alpha$$

and

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leqslant \alpha \frac{|1 - z|^2}{1 - |z|^2} \tag{4}$$

for every z, $|z| < 1$.

We now state and prove our

THEOREM. *Let $P(z)$ be a polynomial of degree $n (\geqslant 2)$ of the form*

$$\begin{aligned} P(z) &= (z - 1)^k Q(z) \qquad (1 \leqslant k < n), \\ Q(z) &= \prod_{i=1}^{n-k} (z - \alpha_i), \qquad |\alpha_i| \leqslant 1, \qquad \alpha_i \neq 1. \end{aligned} \tag{5}$$

Let $L(z)$ and $\beta(z)$ be defined by (1) *and* (1′). *Then*

(a) *If $z \in D(1/(1-c), c/(1-c))$, $0 < c < 1$, or if $z \notin D(1/(1-c), c/(c-1))$, $c > 1$, then*

$$\beta(z) \in D\left(\frac{k}{k + (n-k)c}, \frac{(n-k)c}{k + (n-k)c}\right);$$

(b) *If* $\operatorname{Re} z > 1$, *then*

$$\beta(z) \in D\left(\frac{k}{n}, 1 - \frac{k}{n}\right).$$

Conclusions (a) *and* (b) *may be conveniently summarized as follows*:

(a′) *If* $z \in D(1/(1+c), c/(1+c))$, *then*

$$\beta\left(\frac{1}{z}\right) \in D\left(\frac{k}{k+(n-k)\,c}, \frac{(n-k)\,c}{k+(n-k)\,c}\right),$$

for any positive number c.

Proof. By (1), (1′) and (5), we have

$$L(z) = \frac{k}{z-1} + \frac{Q'(z)}{Q(z)} = \frac{k}{z-1} + \frac{n-k}{z-\alpha(z)} = \frac{n}{z-\beta(z)},$$

for $|z| > 1$, where $\alpha(z)$ and $\beta(z)$ are holomorphic and of modulus less than one in the exterior of the closed unit disk.

Denoting $\alpha^*(z) = \alpha(1/z)$, $\beta^*(z) = \beta(1/z)$, we can rewrite the last equality in the form

$$\frac{1}{z} L\left(\frac{1}{z}\right) = \frac{k}{1-z} + \frac{n-k}{1-z\alpha^*(z)} = \frac{n}{1-z\beta^*(z)} \tag{6}$$

for $|z| < 1$, where

$$\alpha^*(z) = \frac{1}{z} - (n-k)\,\frac{Q(1/z)}{Q'(1/z)}. \tag{7}$$

It follows by (6) that

$$\beta^*(z) = \frac{1}{z}\left[1 - \frac{n(1-z)(1-z\alpha^*(z))}{n-kz\alpha^*(z)-(n-k)\,z}\right]. \tag{8}$$

Applying (7) and (8) it is not difficult to show that $\beta^*(z)$ satisfies the conditions of Julia's theorem. Indeed, by (7) and the well-known Gauss–Lucas theorem on the location of the zeros of the derivative of a polynomial, it follows that $\alpha^*(z)$ is analytic at the point $z = 1$ and $\alpha^*(1) \neq 1$. Since $|\beta(z)| < 1$ for $|z| < 1$, conditions (2) and (3) will be proved if we can show that

$$\lim_{x \to 1^-} \frac{1-\beta^*(x)}{1-x} = \alpha$$

exists and is finite. To evaluate this limit, set $y = 1 - x$ and expand the numerator and the denominator in Taylor series about 0. A straightforward calculation leads to the value

$$\alpha = \frac{n}{k} - 1.$$

Applying inequality (4), we have

$$\frac{|1-\beta^*(z)|^2}{1-|\beta^*(z)|^2} \leqslant \left(\frac{n}{k}-1\right)\frac{|1-z|^2}{1-|z|^2}. \tag{9}$$

Now, for arbitrary $c > 0$, the locus of points z in the complex plane satisfying the inequality $|1 - z|^2 \leqslant c(1 - |z|^2)$ is exactly the set $D(1/(1 + c), c/(1 + c))$. Thus, one obtains from (9) the assertion (a′) of the theorem which is equivalent to (a) and (b).

Observe that each of the disks of (a′) of our theorem lies in the closed unit disk, contains L, and increases from this single point to $D(0, 1)$ as $c \to \infty$. Here are two simple applications of the theorem:

EXAMPLE 1. Consider the equation

$$P(z) - aP'(z) = 0 \tag{10}$$

where a is a constant and $P(z)$ is as in (5). Obviously every point in the closed unit disk is a possible root of Eq. (10). If, however, $\zeta, |\zeta| > 1$, is a root of (10), then (1) and (1′) imply that

$$\zeta - \beta(\zeta) = na.$$

It follows by our theorem that if

$$\frac{1}{\zeta} \in D\left(\frac{1}{1+c}, \frac{c}{1+c}\right),$$

then

$$\zeta - na \in D\left(\frac{k}{k+(n-k)\,c}, \frac{(n-k)\,c}{k+(n-k)\,c}\right).$$

We have, therefore, the following result:

Every root of Eq. (10) *which lies exterior to the closed unit disk must also lie in the* disk

$$D\left(na + \frac{k}{k+(n-k)\,c_0}, \frac{(n-k)\,c_0}{k+(n-k)\,c_0}\right)$$

where c_0 is the smallest positive number such that

$$\frac{1}{\zeta} \in D\left(\frac{1}{1+c_0}, \frac{c_0}{1+c_0}\right).$$

Of course, one can also give c_0 explicitly if ζ is known.

EXAMPLE 2. Consider the polar derivative $P_1(z)$ of the polynomial $P(z)$ of (5) (see [2, III, Section 13]), defined by

$$P_1(z) = nP(z) + (\zeta - z)\, P'(z).$$

A fundamental property of $P_1(z)$ can be expressed as follows: *If all the zeros of $P(z)$ lie in a circular region C (closed interior or exterior of a circle or closed half-plane) and if Z is a zero of $P_1(z)$, then not both points Z and ζ may lie outside C.* Clearly, if Z is a zero of $P_1(z)$ which lies exterior to the closed unit disk, then Z is also a root of the equation

$$P'(z)/P(z) = n/(z - \zeta).$$

Therefore, by our theorem, we also have that $\zeta = \beta(Z)$. We thus have the following generalization of the above fundamental property of polar derivatives:

Let Z be a zero of the polar derivative $P_1(z)$ of $P(z)$ which lies exterior to the closed unit disk. Let $c_1 > 0$ be the smallest number such that

$$\frac{1}{Z} \in D\left(\frac{1}{1 + c_1}, \frac{c_1}{1 + c_1}\right).$$

Then the "pole" ζ of $P_1(z)$ lies in the disk

$$D\left(\frac{k}{k + (n - k)\, c_1}, \frac{(n - k)\, c_1}{k + (n - k)\, c_1}\right).$$

Remarks. (a) Since the zeros of $P_1(z)$ are invariant under a general linear transformation, the last statement can be modified so as to hold for other circular regions as well.

(b) Further applications of our theorem will be published elsewhere.

REFERENCES

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